ARTICLES

Parametrically driven microparticle in the presence of a stationary zero-mean stochastic source: Model for thermal equilibrium in the Paul trap

Alexander F. Izmailov,[1] Stephen Arnold,[2] and Allan S. Myerson[1]
[1]School of Chemical & Materials Science, Polytechnic University, Six MetroTech Center, 333 Jay Street, Brooklyn, New York 11201
[2]Microparticle Photophysics Laboratory, Polytechnic University, Six MetroTech Center, 333 Jay Street, Brooklyn, New York 11201
(Received 2 November 1993)

An analytical approach is developed to consider confined motion of a charged microparticle within the Paul trap (an electrodynamic levitator trap) in an atmosphere near the standard temperature and pressure. The suggested approach is based on a second-order linear stochastic differential equation which describes dampled microparticle motion subjected to the combined periodic parametric and random external excitations. To solve this equation a new ansatz is developed. This ansatz is a generalization of the Bogoliubov-Krylov decomposition technique, which is usually used to reduce the order of a differential equation. The solution is obtained in the long time imaging limit by applying the Bogoliubov general averaging principle. In spite of the second-order form of the initial stochastic differential equation, the microparticle motion can be understood as a one-dimensional Markov process. Comparison in the long time imaging limit of the calculated data obtained from the analytically derived expression for the standard deviation of confined microparticle stochastic motion with the experimentally obtained data demonstrates asymptotic agreement for regions where the dimensionless parameter κ is much less than 1 ($\kappa \leq 0.005$). Simple extremum analysis of the expression obtained for the standard deviation reveals that for the particular case of a large drag parameter α ($\alpha \gg 8\sqrt{12}$) there is a minimum in the standard deviation which is only α dependent.

PACS number(s): 02.50.−r, 85.70.Nk

I. INTRODUCTION

At present spectroscopy of an electrodynamically confined microparticle is a rapidly growing field. Infrared [1], fluorescence [2,3], Raman [4], and photoemission [5] spectroscopies of such microparticles have been demonstrated. There has been a great deal of attention paid to the dynamical limitations in confinement of individual subatomic charged particles in the Penning traps [6,7] and of atomic ions in the Paul traps [8,9] in vacuum. However, corresponding studies of trapping and levitation of a charged microparticle within the Paul trap [an electromagnetic levitator trap (ELT)] in an atmosphere near standard temperature and pressure (STP) [10,11] have received little attention. Experimental and numerical analysis of stability in trapping was performed in [11,12]. In these papers the numerical analysis of solution stability of the standard Mathieu equation with respect to the measurable experimental parameters was given.

Recently the ELT has been successfully applied to the study of nucleation and crystallization [13,14] phenomena in supersaturated solutions. The ELT confined droplets of supersaturated solutions give a unique opportunity to study homogeneous nucleation and crystallization since there is no container in the ELT experiments. This provides an opportunity to achieve the very high solute supersaturation inside of the containerless levitated droplet.

In Ref. [15] the problem of dynamical confinement of an electrically charged microparticle in the ELT in an atmosphere near STP has received a new level of treatment. It was suggested to include thermally induced fluctuations of the trapped microparticle located near the ELT null point. Furthermore, the effect of these fluctuations on long term imaging was investigated. Thus, the problem of confinement in an atmosphere near STP necessitates a stochastic approach. As a result of the work [15], one can state that the motion of an electrically charged microparticle in the ELT is governed by an external parametric periodic force in the presence of an additional drag force and broadband random external excitation. It was demonstrated in Ref. [15] that the last two forces should always come together due to the fluctuation-dissipation theorem [16].

In spite of the significant interest paid to dynamics of the ELT confined microparticles during the last ten years, there is still no comprehensive analytic solution to the problem. In this paper an attempt to build up such a comprehensive analytic solution is presented. This attempt consists of an analytical description of the damping motion of the ELT confined microparticle in an atmosphere near STP. In the particular case under consideration the confined microparticle damping motion is subjected to the simultaneous action of two external excitations: periodic parametric and broadband random. In Sec. II we will introduce and define an equation describing the confined microparticle stochastic motion. In Sec.

1063-651X/94/50(2)/702(7)/$06.00

III an ansatz will be developed to solve this equation. The ansatz is a generalization of the Bogoliubov-Krylov decomposition technique [17] which is usually used to reduce the differential equation order. The equation solution is obtained in a long time imaging limit (i.e., averaged over many cycles) by applying the Bogoliubov general averaging principle [18]. In Sec. IV a summary and conclusions will be presented. These will include the derivation and analysis of the autocorrelation function and the standard deviation of the microparticle confined stochastic motion. The analysis performed in Sec. IV will be accomplished by comparison to the analytically derived results for standard deviation with the experimentally obtained data.

II. EQUATION FOR THE ELT CONFINED ELECTRICALLY CHARGED MICROPARTICLE

A typical trap for microparticle experiments is shown in Fig. 1. A micrometer-sized particle is charged and injected into the ELT either through a hole in the top electrode or through the side (as shown) using a single particle injector such as an on-demand jet [3]. The ELT consists of three electrodes. The top and bottom electrodes are hyperboloids of revolution spaced by $2z_0$, and the center electrode is a torus having a hyperbolic cross section [9]. The time varying voltage $V_1\cos(\omega t)$ is applied to the torus and, relative to the top and bottom of the electrode's interior, a nearly perfect oscillating quadrupole potential $\Phi(\rho,z;t)$ is produced:

$$\Phi_{ac}(\rho,z;t)=V_1\left[\frac{1}{2}-\frac{2z^2-\rho^2}{4z_0^2}\right]\cos(\omega t)\ , \quad (1)$$

where ρ is the cylindrical coordinate ($\rho^2=x^2+y^2$). In addition, a constant voltage V_{dc} is divided equally between the top and center, and center and bottom electrodes in order to produce a static interior potential $\Phi_{dc}(\rho,z)$. This potential balances gravity at the ELT null point.

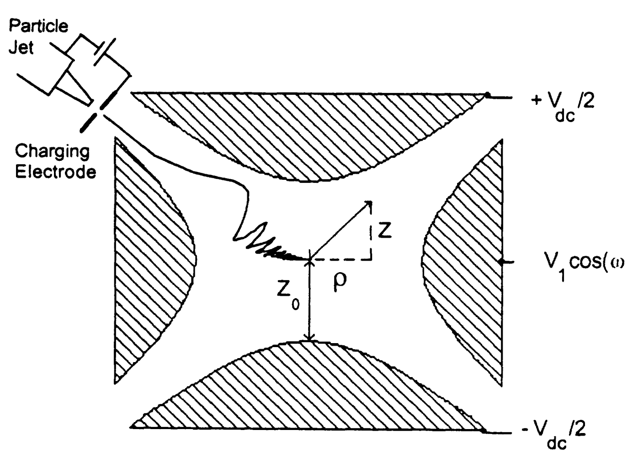

FIG. 1. Hyperbolic electrodynamic levitator trap (Paul trap) for microparticle experiments. The trajectory shown is the simulation of the electrically charged microparticle injected into the N_2 atmosphere at STP. This simulation corresponds to the following dimensionless drag and drive parameters: $\alpha=3.0$ and $\beta=3.28$, as they are defined in Eq. (6).

A spherical electrically charged microparticle injected in a gaseous atmosphere is pulled to the ELT center by alternating gradient forces as illustrated in Fig. 1. The ELT used in our experiments is modified to eliminate stray static fields at its null point. Therefore, in the absence of fluctuations in the microparticle location and with the microparticle weight balanced, the dynamic equation describing microparticle motion around the ELT null point can be given by the following expression (second Newton's law):

$$m\frac{d^2\mathbf{r}}{dt^2}+f\frac{d\mathbf{r}}{dt}-q\mathbf{E}_{ac}=0\ , \quad (2)$$

where m and q are the microparticle mass and electric charge, respectively, $\mathbf{r}$ is the particle radial position vector, and f is the Stokes drag coefficient. In our experiments we are principally interested in viewing the particle from the top (x-y plane) or the side (e.g., z-x plane) directions. Equation (2) easily separates into similar independent equations along the z axis and is perpendicular to it. We restrict our interest for the moment to the motion along the z axis for which the following dynamic equation can be derived from expression (1) and Eq. (2):

$$m\frac{d^2z(t)}{dt^2}+f\frac{dz(t)}{dt}-\frac{qV_1}{z_0^2}\cos(\omega t)z(t)=0\ . \quad (3)$$

This equation has the form of the Mathieu equation with damping. In a region of stability, $z(t)$ damps exponentially [19]. This means that the particle eventually settles to the ELT center and does not move. However, it is easily observed in experiments that the particle does not settle to rest but randomly moves around the ELT null point [15]. This means that Eq. (3) is not complete and should be supplemented with a random source term that describes fluctuations in the microparticle location around the ELT null point. One can come to the same conclusion from the fluctuation-dissipation theorem [16] which states that the introduction of a dissipative force (i.e., of drag) requires simultaneous introduction of the corresponding random force $R(t)$ in the form of a source term. Thus, Eq. (3) should be rewritten in the form

$$m\frac{d^2z(t)}{dt^2}+f\frac{dz(t)}{dt}-\frac{qV_1}{z_0^2}\cos(\omega t)z(t)=R(t)\ . \quad (4)$$

This equation defines time evolution the confined microparticle vertical coordinate $z=z(t)$ as a random process subjected to periodic parametric excitation [15]. Let us consider in this paper the particular case when the random process $z(t)$ is the Markov process (the process without an after effect). From this it follows that the random force $R(t)$ can also be defined as the Markov process. Usually, for simplicity, the additional restriction that the Markov process $R(t)$ is the stationary zero-mean δ-correlated one is imposed. Therefore, the process $R(t)$ acquires a sense of the stationary zero-mean white noise which can be completely characterized by the following one-time probability $P_1(t)$ and two-time conditional probability $P_2(t_1,t_2)$ (i.e., by the fluctuation-dissipation relations),

$$P_1(t)=\langle R(t)\rangle=0 \, ,$$
$$P_2(t_1,t_2)=\langle R(t_1)R(t_2)\rangle=\sigma^2\delta(t_1-t_2) \, , \tag{5}$$

where $\langle \ \rangle$ denotes the ensemble average and $\sigma^2=2k_BTf$ is the white noise variance [16].

Introducing the new variable $z(\tau)=z[\omega t/(2\Omega)]$, Eq. (4) can be rewritten in the following form:

$$\frac{d^2z'(\tau)}{d\tau^2}+\alpha\Omega\frac{dz'(\tau)}{d\tau}-\beta\Omega^2\cos(2\Omega\tau)z'(\tau)=\Omega^2F(\tau) \, , \tag{6}$$

$$\alpha=\frac{2f}{m\omega}, \quad \beta=\frac{4qV_1}{m(z_0\omega)^2} \, ,$$

$$z'(\tau)=z\left[\frac{2\Omega\tau}{\omega}\right]=z(t), \quad f(\tau)=\frac{4}{m\omega^2}R\left[\frac{2\Omega\tau}{\omega}\right] .$$

In Eq. (6) we have introduced the dimensionless drag α and drive β parameters together with the new dimensionless independent variable $\tau=\omega t/(2\Omega)$, where Ω is the arbitrary dimensionless small parameter.

III. SOLUTION OF THE EQUATION FOR THE ELT CONFINED ELECTRICALLY CHARGED MICROPARTICLE IN THE LONG TIME IMAGING LIMIT

We will look for a solution of Eq. (6) in the form

$$z'(\tau)=z_0(\tau)+z_1(\tau)\cos(2\Omega\tau)+z_2(\tau)\sin(2\Omega\tau) \, . \tag{7}$$

Let us assume that there is the following expression for the derivative $dz'(\tau)/d\tau$:

$$\frac{dz'(\tau)}{d\tau}=\frac{dz_0(\tau)}{d\tau}+2\Omega[-z_1(\tau)\sin(2\Omega\tau)+z_2(\tau)\cos(2\Omega\tau)] \, . \tag{8}$$

This expression for $dz'(\tau)/d\tau$ can be correct if and only if the following condition is satisfied:

$$\frac{dz_1(\tau)}{d\tau}\cos(2\Omega\tau)+\frac{dz_2(\tau)}{d\tau}\sin(2\Omega\tau)=0 \, . \tag{9}$$

Now let us substitute expressions (8) and (9) into Eq. (6):

$$\frac{d^2z_0(\tau)}{d\tau^2}+\alpha\Omega\frac{dz_0(\tau)}{d\tau}+2\Omega\left\{-\left[\alpha\Omega z_1(\tau)+\frac{dz_1(\tau)}{d\tau}\right]\sin(2\Omega\tau)+\left[\alpha\Omega z_2(\tau)+\frac{dz_2(\tau)}{d\tau}\right]\cos(2\Omega\tau)\right\}$$
$$-\Omega^2\{\beta z_0(\tau)\cos(2\Omega\tau)+[z_1(\tau)\cos(2\Omega\tau)+z_2(\tau)\sin(2\Omega\tau)][4+\beta\cos(2\Omega\tau)]\}=\Omega^2F(\tau) \, . \tag{10}$$

It is understandable that the nonoscillating and oscillating terms in the above equation (10) should be compensated independently of each other. This necessitates dividing Eq. (10) into two equations containing the nonoscillatory and oscillatory terms, respectively, according to the scheme:

$$\frac{d^2z_0(\tau)}{d\tau^2}+\alpha\Omega\frac{dz_0(\tau)}{d\tau}-\tfrac{1}{2}\beta\Omega^2z_1(\tau)=\Omega^2F(\tau) \, , \tag{11a}$$

$$2\Omega\left\{-\left[\alpha\Omega z_1(\tau)+\frac{dz_1(\tau}{d\tau}\right]\sin(2\Omega\tau)+\left[\alpha\Omega z_2(\tau)+\frac{dz_2(\tau)}{d\tau}\right]\cos(2\Omega\tau)\right\}$$
$$-\Omega^2\{z_1(\tau)[4\cos(2\Omega\tau)+\tfrac{1}{2}\beta\cos(4\Omega\tau)]+z_2(\tau)[4\sin(2\Omega\tau)+\tfrac{1}{2}\beta\sin(4\Omega\tau)]\}=\beta\Omega^2\cos(2\Omega\tau)z_0(\tau) \, . \tag{11b}$$

Following the procedure first suggested by Bogoliubov and Krylov [17], Eq. (11b) can be split into two equations for the derivatives $dz_1(\tau)/d\tau$ and $dz_2(\tau)/d\tau$ in such a way that condition (9) imposed on these derivatives is satisfied:

$$\frac{dz_1(\tau)}{d\tau}=2\Omega\sin(2\Omega\tau)G[z_1(\tau),z_2(\tau);\Omega\tau] \, , \tag{12a}$$

$$\frac{dz_2(\tau)}{d\tau}=-2\Omega\cos(2\Omega\tau)G[z_1(\tau),z_2(\tau);\Omega\tau] \, , \tag{12b}$$

where

$$G[z_1(\tau),z_2(\tau);\Omega\tau]=-z_1(\tau)\left[\cos(2\Omega\tau)+\frac{\alpha}{2}\sin(2\Omega\tau)+\frac{\beta}{8}\cos(4\Omega\tau)\right]$$
$$-z_2(\tau)\left[\sin(2\Omega\tau)-\frac{\alpha}{2}\cos(2\Omega\tau)+\frac{\beta}{8}\sin(4\Omega\tau)\right]-\frac{\beta}{4}z_0(\tau)\cos(2\Omega\tau) \, .$$

It is worthwhile to note that the second-order differential stochastic equation (11a) together with the first-order ordinary differential equations (12a) and (12b) represent an exact decomposition of the initial equation (6). This decomposition constitutes our ansatz for the solution of Eq. (6) and is a generalization of the Bogoliubov-Krylov decomposition procedure [17] which is usually used in nonlinear problems to reduce the differential equation order.

We are interested in a solution of Eqs. (11a), (12a), and (12b) corresponding to the long time imaging limit (i.e., averaged over many cycles). To find such a solution one can use the general averaging principle by Bogoliubov [18]. According to this principle, for $2\Omega \to 0$ the solutions $z_1(\tau)$ and $z_2(\tau)$ of system (12a) and (12b) in the interval of the time of order $O(1/(2\Omega))$ can be approximated uniformly and arbitrary close by solutions of the following system of equations:

$$\frac{dz_1(\tau)}{d\tau} = 2\Omega G_{1,\mathrm{av}}[z_1(\tau), z_2(\tau)] , \tag{13a}$$

where

$$G_{1,\mathrm{av}}[z_1,(\tau),z_2(\tau)] = \lim_{T\to 0} \frac{1}{T} \int_0^T d\xi \sin\xi G[z_1(\tau),z_2(\tau);\xi] ,$$

$$\frac{dz_2(\tau)}{d\tau} = -2\Omega G_{2,\mathrm{av}}[z_1(\tau), z_2(\tau)] , \tag{13b}$$

where

$$G_{1,\mathrm{av}}[z_1(\tau),z_2(\tau)] = \lim_{T\to 0} \frac{1}{T} \int_0^T d\xi \cos\xi G[z_1(\tau),z_2(\tau);\xi] .$$

In these equations $T = \pi/\Omega \to \infty$ with $\Omega \to 0$. The above general averaging principle by Bogoliubov constitutes the widely used averaging principle in nonlinear mechanics [17,18]. This principle was proved by Bogoliubov under quite generally two main assumptions: functions $\sin(2\Omega\tau)G[z_1(\tau),z_2(\tau);\Omega\tau]$ and $\cos(2\Omega\tau)G[z_1(\tau),z_2(\tau);\Omega\tau]$; (a) should be bounded and (b) should satisfy the Lipschitz condition [16,17] with a constant which is independent of $\Omega\tau$. (Different approaches to prove the Bogoliubov general averaging principle can be found in [20,21].) Together the conditions (a) and (b) mean that there exists the following inequality:

$$|G[z_1(\tau),z_2(\tau);\Omega\tau] - G[z_1(\tau'),z_2(\tau');\Omega\tau']| \le (2\Omega)^2 \max\left[1, \frac{\alpha}{2}, \frac{\beta}{4}\right] \sum_{n=1}^{2} |z_n(\tau) - z_n(\tau')| ,$$

where the notation | | means the absolute value and the function $\max(1,\alpha/2,\beta/4)$ makes a choice for the maximum between 1, $\alpha/2$, and $\beta/4$.

Thus instead of a system of equations [(12a) and (12b)] we obtain for the limiting case of long time imaging the following system of truncated equations:

$$\frac{dz_1(\tau)}{d\tau} = -2\Omega\left[\frac{\alpha}{4} z_1(\tau) + \tfrac{1}{2} z_2(\tau)\right] , \tag{14a}$$

$$\frac{dz_2(\tau)}{d\tau} = -2\Omega\left[\frac{\alpha}{4} z_2(\tau) - \tfrac{1}{2} z_1(\tau) - \frac{\beta}{8} z_0(\tau)\right] . \tag{14b}$$

These two first-order ordinary differential equations together with the second-order stochastic differential equation (8a) provide a description of the ELT confined electrically charged microparticle for the limiting case of long time imaging. It is straightforward to obtain solutions for the system of equations [(8a), (14a), and (14b)] in the form of the Fourier integrals,

$$z_0(\tau) = \frac{\Omega^2}{2\pi} \int_{-\infty}^{\infty} d\xi\, e^{i\xi\tau} \bar{F}(\xi) \frac{A(\xi) + d_1}{A^2(\xi) + A(\xi)d_1 + d_2} , \tag{15a}$$

$$z_1(\tau) = -\frac{\beta\Omega^4}{8\pi} \int_{-\infty}^{\infty} d\xi\, e^{i\xi\tau} \bar{F}(\xi) \frac{1}{A^2(\xi) + A(\xi)d_1 + d_2} , \tag{15b}$$

$$z_2(\tau) = \frac{\beta\Omega^3}{8\pi} \int_{-\infty}^{\infty} d\xi\, e^{i\xi\tau} \bar{F}(\xi) \frac{i\xi + \frac{\alpha\Omega}{2}}{A^2(\xi) + A(\xi)d_1 + d_2} , \tag{15c}$$

where

$$A(\xi) = -\xi^2 + i\alpha\Omega\xi ,$$

$$d_1 = \frac{\Omega^2}{4}(\alpha^2 + 4) ,$$

$$d_2 = \tfrac{1}{8}(\beta\Omega^2)^2 .$$

In the above expressions (15a)–(15c) $\bar{F}(\xi)$ is the random function $F(\tau)$ Fourier image. As it follows from expressions (5) the fluctuation-dissipation relations for the random function $F(\tau)$ Fourier images $F(\xi)$ can be given in the form

$$\langle \bar{F}(\xi) \rangle = 0 ,$$
$$\langle \bar{F}(\xi_1)\bar{F}(\xi_2) \rangle = \frac{2\pi\Gamma}{\Omega} \delta(\xi_1 + \xi_2) , \tag{16}$$

where $\Gamma = 8\sigma^2/(m^2\omega^3)$.

IV. SUMMARY: AUTOCORRELATION FUNCTION AND STANDARD DEVIATION OF THE ELT CONFINED ELECTRICALLY CHARGED MICROPARTICLE IN THE LONG TIME IMAGING LIMIT

The autocorrelation function $W'(\tau,\tau+\rho)$ which defines the conditional probability for the Markov process $z'(\tau)$, introduced in Sec. II, can be given by the expression

$$W'(\tau,\tau+\rho) = \langle z'(\tau) z'(\tau+\rho) \rangle . \tag{17}$$

As has been stated in the previous section, we are interested in the limiting case which corresponds to long time imaging. It follows from the Bogoliubov general averaging principle [17,18] that in this limiting case the autocorrelation function $W'(\tau,\tau+\rho)$ can be approximated uniformly and arbitrary closely by its average $W'(\rho)$ over the time interval $[0,T]$:

$$W'(\rho) = \lim_{T\to 0} \frac{1}{T_0} \int_0^T d\tau\, W'(\tau,\tau+\rho) . \tag{18}$$

Carrying out the calculations prescribed by the relations (17) and (18) in the closed form is a straightforward but quite cumbersome procedure. To simplify calculations and the final expression for the function $W'(\rho)$ it is of use to introduce the following small parameter κ:

$$\kappa=\frac{8\beta^2}{(\alpha^2+4)^2} .$$

This parameter is much less than one when $\beta \ll \sqrt{2}+\alpha^2/2\sqrt{2}$. Substituting the expression for $z'(\tau)$ obtained in the previous section [see relations (15a)–(15c)] into Eq. (18) for the function $W'(\rho)$ and carrying out the calculations prescribed above one can obtain the averaged autocorrelation function $W(t)=W'(\omega t/2\Omega)=W'(\tau)$ as the following expansion with respect to the small parameter κ:

$$W(t)=\frac{\Gamma}{2[\lambda_1^2(\alpha,\kappa)-\lambda_2^2(\alpha,\kappa)]}\times[Q_0(\alpha,\kappa;\omega t)+Q_1(\alpha,\kappa;\omega t)\cos(\omega t)+Q_2(\alpha,\kappa;\omega t)\sin(\omega t)] , \tag{19}$$

where

$$Q_0(\alpha,\kappa;\omega t)=\frac{\Gamma}{\alpha^3 A}\left[\left[\frac{1}{\kappa}+A+\tfrac{3}{2}A^2\kappa\right]e^{-(\omega t/2)\lambda_2(\alpha,\kappa)}-\frac{A}{2}(1+\tfrac{3}{2}A\kappa)e^{-(\omega t/2)\lambda_1(\alpha,\kappa)}\right]+O(\kappa^2) ,$$

$$Q_1(\alpha,\kappa;\omega t)=\frac{\Gamma}{8\alpha}\left[\left[1+\frac{2A+1}{2}\kappa\right]e^{-(\omega t/2)\lambda_2(\alpha,\kappa)}-\frac{2A-1}{4}\kappa e^{-(\omega t/2)\lambda_1(\alpha,\kappa)}\right]+O(\kappa^2) ,$$

$$Q_2(\alpha,\kappa;\omega t)=\frac{\Gamma\kappa}{16\alpha^2}(e^{-(\omega t/2)\lambda_1(\alpha,\kappa)}-e^{-(\omega t/2)\lambda_1(\alpha,\kappa)})+O(\kappa^2) .$$

The parameters introduced above, $\lambda_1(\alpha,\kappa)$ and $\lambda_2(\alpha,\kappa)$, have the form

$$\lambda_1(\alpha,\kappa)=\alpha\left[1-\frac{A}{2}\kappa\right]+O(\kappa^2) ,$$

$$\lambda_2(\alpha,\kappa)=\frac{\alpha A}{2}\kappa+O(\kappa^2) ,$$

where $A=(\alpha^2+4)/(8\alpha^2)$.

It is straightforward to obtain the second moment $W(0)$ of the microparticle confined stochastic motion within the ELT,

$$W(0)=\frac{\Gamma}{2[\lambda_1^2(\alpha,\kappa)-\lambda_2^2(\alpha,\kappa)]}\times[Q_0(\alpha,\kappa;0)+Q_1(\alpha,\kappa;0)] , \tag{20}$$

where the expressions for $Q_1(\alpha,\kappa;0)$ and $Q_2(\alpha,\kappa;0)$ are given by the relation (19) definitions. The standard deviation Σ of this motion is related to the second moment $W(0)$ as $\Sigma^2=W(0)$.

A simple analysis of expression (20) leads to the following conclusions:

(a) The function $W(0)$ grows as $1/\kappa$ when κ tends to zero. This growth occurs either as β^2 approaches zero or when α^2 goes to infinity in such a way that their particular combination given by the parameter κ remains much less than one. This continues until the nearest region of instability is reached and microparticle is not confined anymore.

(b) The function $W(0)$ slowly grows as κ when the small parameter κ tends to one. It is worthwhile to note that the small parameter κ can go to one either by increasing β^2 or by decreasing α^2.

(c) There is the well defined minimum $W_{\min}(0)$ of the second moment (square of standard deviation) $W(0)$ with respect to the parameter κ.

(d) For the case of the large and fixed drag parameter α ($\alpha_2 \gg 4$) expressions for the second moment minimum $W_{\min}(0)$ and corresponding to this minimum drive parameter $\beta=\beta_{\min}$ acquire the very simple forms. Their dependences on the drag parameter α in this particular case are given by the following simple expressions:

$$W_{\min}(0)=\frac{\Gamma}{8\alpha}\left[1+\frac{4\sqrt{13}}{\alpha}+O\left[\frac{1}{\alpha^2}\right]\right] ,$$

$$\beta_{\min}=\frac{2\alpha^{3/2}}{(13)^{1/4}}\left[1+O\left[\frac{1}{\alpha}\right]\right] . \tag{21}$$

Relations (21) give the interesting possibility of finding the minimum possible second moment $W_{\min}(0)$ and the corresponding drive parameter $\beta_{\min}$ for the specific value of the drag parameter α when two strong inequalities,

$$\alpha^2 \gg 4, \quad \kappa|_{\beta=\beta_{\min}} \ll 1$$

are satisfied simultaneously. This takes place when $\alpha \gg 8\sqrt{12}$. It is noteworthy that, as follows from (21), the driving force increase does not necessarily lead to a reduction of the standard deviation Σ which has the well defined minimum $\Sigma_{\min}$ at $\beta=\beta_{\min}$. More complicated expressions for $\Sigma_{\min}$, and for the corresponding $\beta_{\min}$ for the other, free of the restrictive condition $\alpha \gg 8\sqrt{12}$, cases can be also found by analyzing expression (20). The presence of the minimum $\Sigma_{\min}=W_{\min}^{1/2}(0)$ in the standard deviation Σ dependence on the drag parameter α gives a unique possibility to improve spectroscopic measurements of the ELT confined electrically charged microparticle in an atmosphere near STP. Thus by varying the parameter α (for example, by changing the microparticle size) one can considerably reduce the thermal noise effect on motion of the ELT confined microparticle.

Figure 2 is the comparative plot of the analytically obtained standard deviation Σ_y along the y axis and the experimentally obtained $\Sigma_{y,\mathrm{expt}}$ data. Analytical results for the microparticle confined stochastic motion along the y axis can be easily restored from the same results obtained with respect to the z axis. The only difference in the analytical description of these motions is that the drive

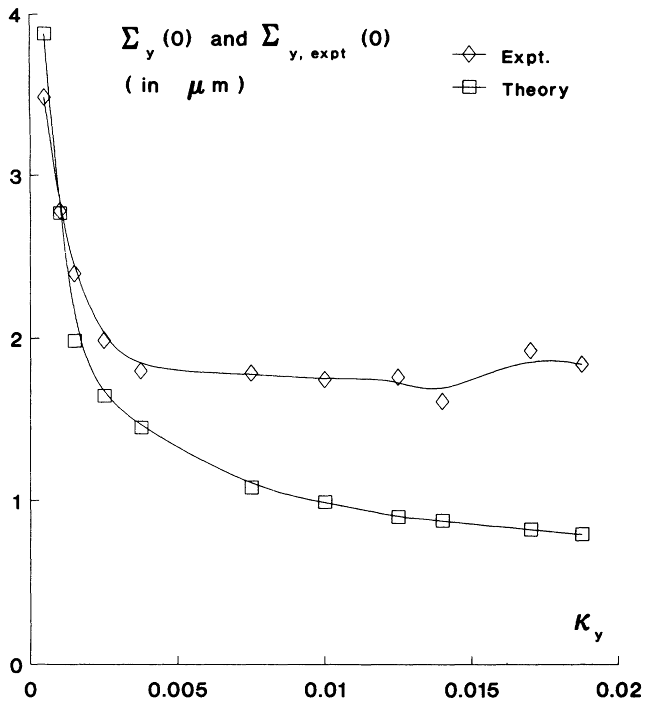

FIG. 2. Comparative plot of the analytically derived Σ_y and experimentally obtained $\Sigma_{y,\mathrm{expt}}$ results for the standard deviation of the microparticle confined stochastic motion along the y axis $[\kappa_y=8\beta_y^2/(\alpha^2+4)^2=2\beta_z^2/(\alpha^2+4)^2]$. The comparison has been performed for the relatively large value of the drag parameter $\alpha=45.37\gg 8\sqrt{12}$ in an atmosphere near STP (temperature is 294.0 K).

parameter β_y along the y axis is half of the same parameter $\beta=\beta_z$ along the z axis [$\beta_y=\beta_z/2$, see expression (1)]. This comparison has been performed for the relatively large value of the drag parameter $\alpha=45.37\gg 8\sqrt{12}$ in an atmosphere near STP (the temperature was 294.0 K). There is a fine coincidence between the experimentally obtained data and the analytical results for small values of the parameter κ ($\kappa\leq 0.005$). However, the further increase of κ ($0.005\leq\kappa\leq 1$) leads to a considerable discrepancy up to 600% between experimental and analytical results. To address and resolve this problem the following model equation is suggested:

$$m\frac{d^2z(t)}{dt^2}+f\frac{dz(t)}{dt}-\frac{qV_1}{z_0^2}\cos(\omega t+\phi)z(t)=R(t)\ . \tag{22}$$

In Ref. [15] Eq. (4) was substituted by Eq. (22). In this equation the quantity ϕ is the random initial phase. Introduction of this phase is physically justified since it accounts for the randomly occurring initial collisions between microparticle and molecules of the ELT atmosphere. All numerical results obtained in [15] are based on the Green function $h_\phi(\omega t/2,\omega t'/2)\equiv h_\phi(\tau,\tau')$ associated with Eq. (22). It has been concluded in [15] that an assumption concerning the time-shift invariance of the Green function $h_\phi(\tau,\tau')$ can be approximately justified in the long time imaging limit. Having this assumption in mind it is straightforward to demonstrate that in terms of the Green function $h_\phi(\tau,\tau')$ the second moment $W(0)$ (standard deviation squared) of the microparticle stochastic motion can be represented in the form,

$$W(0)=\frac{8\sigma^2}{m^2\omega^3}\left\langle\int_0^\infty d\tau h_\phi^2(\tau,0)\right\rangle_\phi\ . \tag{23}$$

In this expression $\langle\ \rangle_\phi$ denotes averaging over the random initial phase ϕ. Numerical analysis of the standard deviation $\Sigma=W^{1/2}(0)$ given by expression (23) has demonstrated much better agreement with experiment [15] in the region where the current theory shows a discrepancy. This difference is surprising since one does not expect the initial random collision to have a long term influence on the microparticle motion. Thus, an analytical solution for the model including a random initial phase is required together with further investigations of the present model.

Comparison between analytical results obtained in this paper and experimental data for the standard deviation has been performed for large values of the drag α and drive β parameters ($\beta\geq 51.56$, $\alpha=45.37$ for $\kappa\geq 0.005$). It is quite likely that for such large values of the parameters α and β energy dissipation in the system under consideration is nonlinear. Since such an effect may be anticipated in future experiments and almost certainly exists to some extent in experiments at these extremes it is useful to elaborate on the form which the nonlinear theory will take. The nonlinear energy dissipation means that the linear second-order stochastic differential equation (4) describing stochastic motion of the ELT confined electrically charged microparticle should be substituted by the nonlinear equation of the following kind:

$$m\frac{d^2z(t)}{dt^2}+f_1\frac{dz(t)}{dt}U\left[z^2(t)+f_2\left[\frac{dz(t)}{dt}\right]^2\right]-\frac{qV_1}{z_0^2}\cos(\omega t)z(t)=R(t)\ .$$

In this equation the function $U\{z^2(t)+f_2[dz(t)/dt]^2\}$ is continuous and depends only on one variable $z^2(t)+f_2[dz(t)/dt]^2$, f_1 and f_2 are the corresponding drag parameters. Such types of functions are widely used to characterize energy dissipation in the nonlinear mechanical systems subjected to internal and external excitations [16,17]. The model of the ELT confined electrically charged microparticle, the stochastic motion of which is described by the nonlinear equation given above is at present under consideration and will be reported in later work.

Thus, in this paper an attempt to develop a theoretical approach describing the ELT confined stochastic motion of the electrically charged microparticle is performed in the long time imaging limit for the particle case when (a) energy dissipation in the system microparticle plus atmosphere is linear and (b) the random source process $R(t)$ corresponding to thermal noise is the stationary zero-mean white noise. It has turned out from the ansatz suggested in this paper [see Eq. (11a) and expressions (14a) and (14b)] that for the particular case of long time imaging the microparticle confined motion can be understood over a specific range as the one-dimensional stationary zero-mean Markov process. This takes place in spite of the second order of the initial stochastic differential equation (4).

ACKNOWLEDGMENT

The authors gratefully acknowledge this support of the National Science Foundation (NSF Grant Nos. ATM-89-05871 and CTS-90-20233).

[1] S. Arnold, M. Neuman, and A. B. Pluchino, Opt. Lett. **9**, 4 (1984).
[2] L. M. Folan, S. Arnold, and S. D. Druger, Chem. Phys. Lett. **118**, 322 (1985).
[3] S. Arnold and L. M. Folan, Rev. Sci. Instrum. **57**, 2250 (1986).
[4] R. E. Preston, T. R. Lettieri, and H. G. Semerjian, Langmuir **1**, 365 (1985).
[5] S. Arnold and N. Hessel, Rev. Sci. Instrum. **56**, 2066 (1985).
[6] D. Wineland, P. Ekstrom, and H. G. Dehmelt, Phys. Rev. Lett. **31**, 1297 (1973).
[7] L. S. Brown and G. Gabrielse, Rev. Mod. Phys. **58**, 233 (1986).
[8] H. G. Dehmelt, in *Advances in Atomic and Molecular Physics* (Academic, New York, 1967), Vol. 3.
[9] W. Paul, Rev. Mod. Phys. **62**, 531 (1990).
[10] H. Winter and H. W. Ortjohann, Am. J. Phys. **59**, 807 (1991).
[11] R. F. Weurker, H. Shelton, and R. V. Langmuir, J. Appl. Phys. **30**, 349 (1959).
[12] E. J. Davis, P. Ravindran, and A. K. Ray, Adv. Colloid Interface Sci. **15**, 1 (1981).
[13] A. S. Myerson, H. S. Na, A. F. Izmailov, and S. Arnold, in *Proceedings of the 12th Symposium on Industrial Crystallization 1,* edited by Z. H. Rojkowski (Warsaw, 1993), pp. 3-013.
[14] H. S. Na, S. Arnold, and A. S. Myerson, J. Cryst. Growth **139**, 104 (1994).
[15] S. Arnold, L. M. Folan, and A. Korn, J. Appl. Phys. **74**, 4291 (1993).
[16] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics* (Pergamon, London, 1986), Vol. 5.
[17] N. N. Bogoliubov and Y. A. Mitropolsky, *Asymptotic Methods in the Theory of Non-Linear Oscillations* (Gordon and Breach, New York, 1961).
[18] N. N. Bogoliubov, *On Certain Statistical Methods in Mathematical Physics* (Ukraine SSR Academy of Sciences, Kiev, 1945), in Russian.
[19] G. Blanch, in *Handbook of Mathematical Functions,* Natl. Bur. Stand. Appl. Math. Ser. No. 55, edited by M. Abramowitz and I. A. Stegun (U.S. CPO, Washington, D.C., 1964).
[20] I. I. Gikhman, Ukr. Mat. Zh. **4**, 215 (1952).
[21] M. A. Krasnoselsky and S. G. Krein, Usp. Mat. Nauk **10** (No. 3), 147 (1955).